Bias on Horizontal Mean-Wind Speed and Variance caused by Turbulence

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Presentation of Main Results

The cup anemometer is characterized by five, wind-speed independent instrument constants. The first two, the *starting speed* $U_\circ$ and the *calibration length* ℓ, determine the calibration, which as a rule is obtained in a wind tunnel with negligible turbulence. The rate of rotation S in such a laminar-flow wind tunnel is for a good standard cup anemometer a linear function of the wind speed U and given by

$$S = \frac{U - U_\circ}{\ell}. \tag{1}$$

Here S is given in rad/s and the calibration length ℓ can, if we neglect the small starting velocity, be interpreted as the length of the column of air which has to pass through the anemometer for the rotor to turn one radian. Another way of stating the calibration equation is to relate a frequency f to the wind speed. If the anemometer generates n pulses every time the rotor has turned one full rotation, (1) implies that

$$U = \frac{2\pi\ell}{n} f + U_\circ. \tag{2}$$

For the Risø model P2546 we have $U_\circ \sim 0.2$ m/s, $\ell \simeq 0.2$ m and $n = 2$ so that $2\pi\ell/n \simeq 0.6$ m.

The third instrument constant is the *distance constant* $\ell_\circ$. The cup anemometer can be looked upon as a first-order filter and the interpretation of $\ell_\circ$ can be perceived by imagining that a constant wind speed U suddenly is changed to $U + \Delta U$. Then $\ell_\circ$ can be interpreted as the length of the column of air which has to pass through the rotor to change the rotation rate from S to $S + (1 - \exp(-1))\Delta S$, which is about 37% of ΔS off its final new equilibrium rate of rotation $S + \Delta S$. This low-pass filtering means that under operating conditions with a mean-wind speed U, the short-period fluctuations will be strongly damped if their periods are smaller than

$$\tau_\circ = \ell_\circ / U. \tag{3}$$

In the case of the Risø model P2546 $\ell_\circ = 1.8$ m $\simeq 9 \times \ell$ (Kristensen & Hansen 2002), which means that the rotor must turn about 1.5 times to adjust to 63% of a wind speed change. In qualitative terms, the cup anemometer cannot pick up any new, important information about changes in the wind speed before the rotor has turned more than one revolution. Consequently, a number of pulses n for one full rotor revolution much larger than one seems unnecessary—an “overkill”.

The last two dimensionless parameters μ_1 and μ_2 specify the angular response $g(\theta)$ of the cup anemometer. For an ideal anemometer $g(\theta)$ is equal to $\cos\theta$. This means that such an instrument measures the wind speed perpendicular to the rotor axis. A real cup anemometer is not ideal and the angular response is specified by

$$g(\theta) = \cos\theta + \mu_1 \sin\theta + \mu_2(1 - \cos\theta) \simeq 1 + \mu_1\theta - (1 - \mu_2)\frac{\theta^2}{2}, \tag{4}$$

where θ in radians is assumed small compared to one. The parameter μ_1 and μ_2 account for the response skewness and response flatness, respectively. We see that for an ideal cup anemometer $\mu_1 = \mu_2 = 0$.

We have now presented the tools for evaluating biases of the mean-wind and the variance due to turbulence. The cup anemometer responds primarily to the fluctuating wind speed in the mean-wind direction. However, since the anemometer is calibrated in a wind tunnel with no fluctuations, we must expect that in a turbulent wind there will be a bias on the measured mean-wind speed proportional to the mean squares of the fluctuations of the three velocity components. By convention we use the symbol u for the fluctuations in the mean-wind direction. The lateral and the vertical components are called v and w, respectively. The analysis presented in the next section shows that the relative bias can be written

$$\frac{\langle s \rangle}{S} = 0.22 \left(1 + \frac{4}{3}\mu_1^2\right) \frac{\langle u^2 \rangle}{U^2} \left(\frac{\ell_\circ}{z}\right)^{2/3} + \frac{\langle v^2 \rangle + \mu_2 \langle w^2 \rangle}{2U^2}, \tag{5}$$

where averaging is indicated by angle brackets. The first term is the "traditional" overspeeding. The cup anemometer responds primarily to not just u, but to the combination $u + \mu_1 w$, and the reason for the bias is the asymmetry of its response to increases and decreases of the wind speed. This means that the signal spends more time on the upper side of the mean than on the lower side. The cup rotor can follow velocity eddies with sizes larger than $\ell_\circ$, but is unable to resolve smaller eddy velocities. This is reflected in the factor $(\ell_\circ/z)^{2/3}$, where z is the instrument height and therefore is a measure of the dominant eddy size. In other words, the bias from $u + \mu_1 w$ is high-pass filtered. We see that the transverse velocity functions enter the expression with all eddy sizes, i.e. unfiltered, and note that if $\mu_2 = 1$ corresponds to a completely flat angular response, so that the vertical fluctuations w in this case enter in the same as the lateral fluctuations v.

For the variance the asymmetric response to increase and decrease in the wind speed is of no consequence. However, the imperfect angular response will cause a bias. The relative bias on the measured variance σ^2 of the velocity in the mean-flow direction is

$$\frac{\sigma^2 - \langle u^2 \rangle}{\langle u^2 \rangle} = \mu_1^2 \frac{\langle w^2 \rangle}{\langle u^2 \rangle} + 2\mu_1 \frac{\langle uw \rangle}{\langle u^2 \rangle}. \tag{6}$$

We have assumed that the cup anemometer is placed in a horizontally homogeneous terrain with its axis pointing towards zenith. If the terrain has a positive or negative slope, the non-ideal angular response (4) will cause a bias of the mean wind velocity measurement compared to that of the ideal response. This is illustrated in Fig. 1, where the effect of μ_1 and μ_2 is displayed.

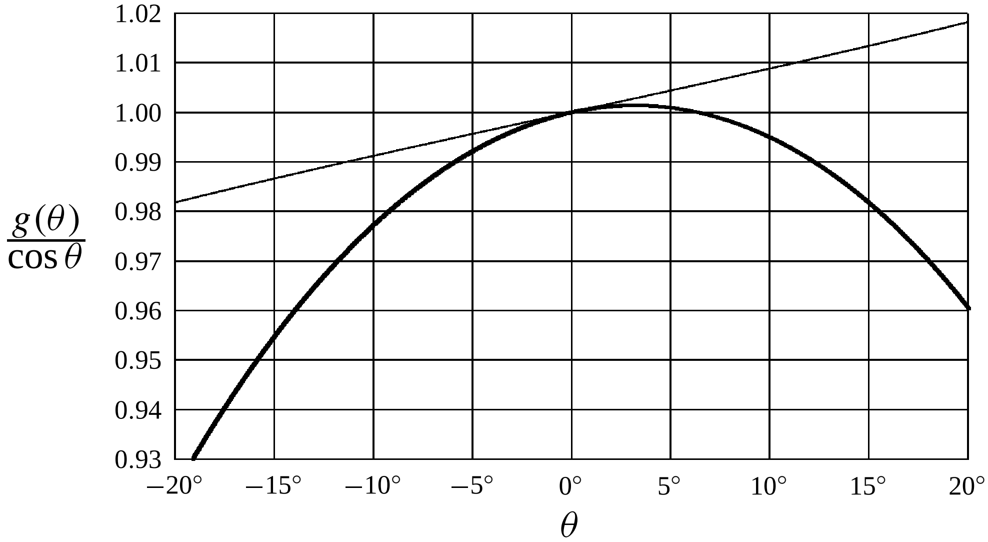

Figure 1: The ratio of the angular response of the Risø model P2546 and the ideal cosine response. The thin line represents exclusively the asymmetric component with $(\mu_1, \mu_2) = (0.05, 0.0)$ while the thick line represents the combined effect of the two angular-response parameters $(\mu_1, \mu_2) = (0.05, -0.9)$.

Bias Determination

From an operational point of view it is possible in many cases to correct for the biases on the mean-wind velocity and the u-variance by employing a wind vane for the determination of the lateral velocity component and/or applying well-established, micrometeorological relations. As an example of practical importance, we consider a situation where the flow is horizontally homogeneous and where the wind speed is so large that the mechanical turbulent mixing dominates that of the convective. This will typically be the case when the sky is overcast and the wind speed is more than a few meters per second. Such an atmosphere has what is called neutrally stratification because no work, neither positive nor negative, is involved in moving an air parcel from one height to another. Panofsky & Dutton (1984) have compiled data from nine field experiments and determined empirical relations between the variances $\langle u^2 \rangle$, $\langle v^2 \rangle$, $\langle w^2 \rangle$, and the covariance $\langle uw \rangle$ for neutral stratification. Since there is always a downward transport of momentum this covariance is always negative, and defines the so-called friction velocity

$$u_* = \sqrt{-\langle uw \rangle}. \tag{7}$$

With this definition the relations can be stated

$$\left\{ \begin{array}{c} \sqrt{\langle u^2 \rangle} \\ \\ \sqrt{\langle v^2 \rangle} \\ \\ \sqrt{\langle w^2 \rangle} \end{array} \right\} = u_* \times \left\{ \begin{array}{c} 2.39 \pm 0.03 \\ \\ 1.92 \pm 0.05 \\ \\ 1.25 \pm 0.03 \end{array} \right\}. \tag{8}$$

Another important relation for the neutral atmosphere is the mean-wind speed as a function of height z:

$$U(z) = \frac{u_*}{\kappa} \ln\left(\frac{z}{z_\circ}\right). \tag{9}$$

Here $\kappa \simeq 0.4$ is the von Kármán constant and $z_\circ$ the surface roughness-length, which depends on the surface characteristics. According to Panofsky & Dutton (1984) $z_\circ$ is about 0.03 m for cut grass and 0.6 m for long grass and crops. Troen & Petersen (1989) and Mortensen et al. (1993) provide operational advice for estimating the roughness length.

The usual situation with a modern datalogging system is that U, $\langle u^2\rangle$, and $\langle v^2\rangle$ are measured by means of a cup anemometer and a wind direction sensor which, for small angles measures v/U. However, the vertical velocity component of the the wind velocity w is seldom measured and $\langle w^2\rangle$ and u_* must therefore be determined indirectly. This is accomplished by calculating first u_* from (9), assuming that z and $z_\circ$ are known. Then the last relation in (8) allows to calculate $\langle w^2\rangle$. All the parameters for determining the bias on the mean-wind (5) and the u-variance (6) are then available.

As (8) and (9) show, it is possible, by means of these two relations to obtain estimates of the two biases (5) and (6) without measuring $\langle u^2\rangle$ and $\langle v^2\rangle$. As an example, let us consider the Risø model P2546 with the distance constant $\ell_\circ = 1.8$ m and the angle parameters $(\mu_1, \mu_2) = (0.05, -0.9)$. This instrument is operated at the altitude $z = 10$ m over a terrain with the roughness length $z_\circ = 0.05$ m and measures the mean wind speed $U = 5$ m/s. We get

$$u_* = \kappa U/\ln(z/z_\circ) = 0.4 \times 5/\ln(10/0.05) \text{ m/s} = 0.37 \text{ m/s}. \tag{10}$$

Thus

$$\left\{\begin{array}{l} U \\ u_* \\ \sqrt{\langle u^2\rangle} \\ \sqrt{\langle v^2\rangle} \\ \sqrt{\langle w^2\rangle} \end{array}\right\} = \left\{\begin{array}{l} 5 \\ 0.37 \\ 0.90 \\ 0.72 \\ 0.47 \end{array}\right\} \text{ m/s}. \tag{11}$$

Inserting into (5) and (6), we see that in this case

$$\frac{\langle s\rangle}{S} = 0.009 \tag{12}$$

and

$$\frac{\sigma^2 - \langle u^2 \rangle}{\langle u^2 \rangle} = -0.31. \tag{13}$$

Elaborate Derivation of Main Results

The most general dynamic equation of motion for the cup anemometer is

$$\dot{\widetilde{s}} \equiv \frac{\mathrm{d}\widetilde{s}}{\mathrm{d}t} = F(\widetilde{s}, \widetilde{h}, \widetilde{w}), \tag{14}$$

where $\widetilde{s}$ is the instantaneous angular velocity of the cup rotor in rad/s,

$$\widetilde{h} = \sqrt{\widetilde{u}^2 + \widetilde{v}^2} \tag{15}$$

the instantaneous wind speed with the components $(\widetilde{u}, \widetilde{v})$ in the rotor plane, and $\widetilde{w}$ the instantaneous wind-velocity component perpendicular to the rotor plane. We imagine that the coordinate system is oriented so that $\widetilde{u}$ is along the mean-wind direction. Figure 2 illustrates a top view and a side view of the cup-anemometer concept.

In a constant wind (wind tunnel) with $(\widetilde{u}, \widetilde{v}, \widetilde{w}) = (U, 0, 0)$ the rotor will have the constant angular velocity $\widetilde{s} = S$. Inserting into (14), we get

$$0 = F(S, U, 0). \tag{16}$$

This equation defines the calibration which, for a good cup anemometer, can be considered linear, i.e.

$$S = \frac{U - U_\circ}{\ell}, \tag{17}$$

where ℓ is the calibration length and $U_\circ$ the so-called starting speed. This quantity is usually small, about 0.2–0.3 m/s, for a good anemometer, and in the following we consider only the normal situation where $U \gg U_\circ$. The approximation

$$\ell S = U \tag{18}$$

is therefore accurate enough for the determination of the bias of the measured mean-wind speed.

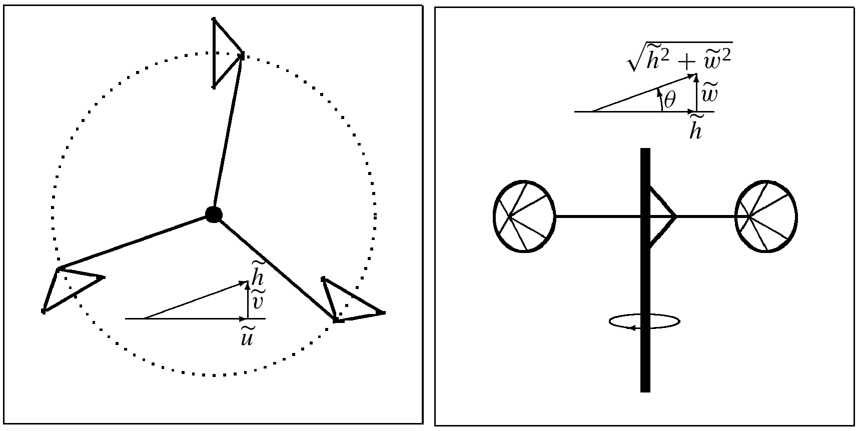

Figure 2: Sketch of a cup anemometer and indications of the wind-velocity components. Left frame: top view. Right frame: side view.

In a turbulent wind field we decompose $\widetilde{s}$ and $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ as follows

$$\left\{ \begin{array}{c} \widetilde{s} \\ \\ \widetilde{u} \\ \\ \widetilde{v} \\ \\ \widetilde{w} \end{array} \right\} = \left\{ \begin{array}{l} S+s \\ \\ U+u \\ \\ v \\ \\ w \end{array} \right\}, \tag{19}$$

where U is equal to the mean $\langle \widetilde{u} \rangle$ and S is given by (17). We must assume that $|s| \ll S$, $|u| \ll U$, $|v| \ll U$, and $|w| \ll U$ for the cup anemometer to operate satisfactorily. An immediate consequence of this assumption is that to second order we have

$$\begin{aligned} \widetilde{h} &= \sqrt{(U+u)^2+v^2} = U\left(1+2\frac{u}{U}+\frac{u^2+v^2}{U^2}\right)^{1/2} \\ &\simeq U\left(1+\frac{1}{2}\left(2\frac{u}{U}+\frac{u^2+v^2}{U^2}\right)+\frac{(1/2)\times(-1/2)}{2}\left(2\frac{u}{U}\right)^2\right) = U+u+\frac{v^2}{2U} \end{aligned} \tag{20}$$

Initially we simplify the model by considering only a horizontally fluctuating wind, i.e. $\widetilde{w} = 0$. The dynamic equation (14) reduces to

$$\dot{\widetilde{s}} = F(\widetilde{s}, \widetilde{h}, 0) \equiv G(\widetilde{s}, \widetilde{h}) \tag{21}$$

and (16) to

$$0 = G(S, U). \tag{22}$$

Expanding (21) to second order in s, u, and v, we get

$$\begin{aligned} \dot{s} &= \underbrace{G(S, U)}_{=0} + \frac{\partial G}{\partial S} s + \frac{\partial G}{\partial U}\left(u + \frac{v^2}{2U}\right) \\ &\quad + \frac{1}{2}\left(\frac{\partial^2 G}{\partial S^2} s^2 + 2\frac{\partial^2 G}{\partial S \partial U} su + \frac{\partial^2 G}{\partial U^2} u^2\right). \end{aligned} \tag{23}$$

All the first and second derivatives are taken at the point (S, U). Neglecting first the second-order terms, we see that (23) becomes the equation for a first-order, linear filter. Since the solution must be finite, we infer that $\partial G/\partial S$ must be negative. In fact, this derivative determines the filter time-constant $\tau_\circ$ by

$$\frac{1}{\tau_\circ} = -\frac{\partial G}{\partial S}. \tag{24}$$

The function $G(S, U)$ is almost entirely determined by the wind drag force on the rotor and is therefore proportional to the air density ϱ multiplied by a second-order polynomial in S and U. We see then, with the aid of (18), that the definition (24) of $\tau_\circ$ implies that this quantity is inversely proportional to $\varrho \times U$. It means that the distance

$$\ell_\circ = U\tau_\circ \tag{25}$$

is independent of the wind speed. We have here introduced the distance constant which, apart from its dependence of the air density, can be considered an instrument constant, in contrast to the time constant $\tau_\circ$.

Differentiating (22), we obtain with the aid of (17)

$$0 = \frac{\partial G}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}U} + \frac{\partial G}{\partial U} = \frac{1}{\ell}\frac{\partial G}{\partial S} + \frac{\partial G}{\partial U}, \tag{26}$$

so that

$$\frac{\partial G}{\partial U} = \frac{1}{\ell\tau_\circ}. \tag{27}$$

The first-order filter function can thus be written

$$\dot{s} + \frac{s}{\tau_\circ} = \frac{u}{\ell \tau_\circ}. \tag{28}$$

It is a useful consequence of (28) that there is a relation between the variance $\langle s^2 \rangle$ and the covariance $\langle su \rangle$. This can be seen by multiplying (28) by s and then taking the average. We use the fact that for the stationary time series s the mean and higher order moments are independent of time, i.e.

$$\left\langle s \, \frac{\mathrm{d}s}{\mathrm{d}t} \right\rangle = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \langle s^2 \rangle = 0. \tag{29}$$

In other words,

$$\langle s^2 \rangle = \frac{\langle su \rangle}{\ell}. \tag{30}$$

Rewriting (23) in the form

$$\tau_\circ \dot{s} + s = \frac{u}{\ell} + \frac{\tau_\circ}{2} \left(\frac{\partial^2 G}{\partial S^2} s^2 + 2 \frac{\partial^2 G}{\partial S \partial U} su + \frac{\partial^2 G}{\partial U^2} u^2 \right) + \frac{v^2}{2U\ell}, \tag{31}$$

we can now determine the bias on the measured wind speed by taking the mean of (31). Since u is the fluctuation around the mean U, $\langle u \rangle$ is zero by definition. The mean $\langle s \rangle$ is of course constant, so that $\langle \dot{s} \rangle = 0$, but $\langle s \rangle$ is not zero, because S is only the mean of the rotation rate $\widetilde{s}$ when there is no turbulence. In fact, the dimensionless quantity $\langle s \rangle / S$ is the relative bias on the measured wind speed. From (31) we get

$$\frac{\langle s \rangle}{S} = \frac{\tau_\circ}{2S} \left\{ \frac{\partial^2 G}{\partial S^2} \langle s^2 \rangle + 2 \frac{\partial^2 G}{\partial S \partial U} \langle su \rangle + \frac{\partial^2 G}{\partial U^2} \langle u^2 \rangle \right\} + \frac{\langle v^2 \rangle}{2U^2}, \tag{32}$$

where in the last term on the right-hand side we have used (18). Just like we obtained (26), we derive a relation between the second derivatives by differentiating (22) twice with respect to U. The result is

$$\frac{1}{\ell^2} \frac{\partial^2 G}{\partial S^2} + \frac{2}{\ell} \frac{\partial^2 G}{\partial S \partial U} + \frac{\partial^2 G}{\partial U^2} = 0. \tag{33}$$

With this equation and (30), we reformulate (32):

$$\frac{\langle s \rangle}{S} = \frac{\ell \tau_\circ}{2U} \frac{\partial^2 G}{\partial U^2} \{ \langle u^2 \rangle - \ell \langle su \rangle \} + \frac{\langle v^2 \rangle}{2U^2}. \tag{34}$$

We need to determine the coefficient $\ell\tau_\circ/(2U) \times \partial^2 G/\partial U^2$. Fortunately, this has been done by Wyngaard et al. (1974) and by Coppin (1982) in rather sophisticated wind-tunnel experiments. Their results are summarized by Kristensen (1998) and for a number of widely different types of cup anemometers they found that within about 10%

$$\frac{\ell\, U\tau_\circ}{2}\frac{\partial^2 G}{\partial U^2} = 1, \tag{35}$$

independent of the mean-wind speed U. We will assume that this instrument constant, which was originally called a_4 by Wyngaard et al. (1974), is one and, consequently, that (34) becomes

$$\frac{\langle s\rangle}{S} = \frac{1}{U^2}\{\langle u^2\rangle - \ell\langle su\rangle\} + \frac{\langle v^2\rangle}{2U^2}. \tag{36}$$

To evaluate $\ell\langle su\rangle$ we need to find out how $s = s(t)$ depends on $u = u(t)$. We find the relation by solving (28). The solution is

$$s(t) = \frac{1}{\ell}\int_0^\infty u(t-\tau)\,\mathrm{e}^{-\tau/\tau_\circ}\,\frac{\mathrm{d}\tau}{\tau_\circ}. \tag{37}$$

This leads to

$$\begin{aligned}
\langle u^2\rangle - \ell\langle su\rangle &= \langle u^2\rangle - \left\langle \int_0^\infty u(t)u(t-\tau)\,\mathrm{e}^{-\tau/\tau_\circ}\,\frac{\mathrm{d}\tau}{\tau_\circ}\right\rangle \\
&= \int_0^\infty \langle u^2 - u(t)u(t-\tau)\rangle\,\mathrm{e}^{-\tau/\tau_\circ}\,\frac{\mathrm{d}\tau}{\tau_\circ} \\
&= \frac{1}{2}\int_0^\infty \left\langle [u(t-\tau)-u(t)]^2\right\rangle\,\mathrm{e}^{-\tau/\tau_\circ}\,\frac{\mathrm{d}\tau}{\tau_\circ}.
\end{aligned} \tag{38}$$

The quantity average $\langle[u(t-\tau)-u(t)]^2\rangle$ is the so-called structure function which, for stationary signals like $u(t)$, depends on the time difference τ and is independent of the absolute time t (Lumley & Panofsky 1964, page 46).

At this point we introduce Taylor's "frozen turbulence" hypothesis (Lumley & Panofsky 1964, Tennekes & Lumley 1978, Panofsky & Dutton 1984, Frisch 1995, Blackadar 1997, Pope 2000, Tsinober 2001). In an informal way it states that all (fast and turbulent) temporal fluctuations at a given point are entirely made up of "stiff" spatial velocity fluctuations which are carried through the point by the mean wind. It means that the above mentioned structure function can be considered a function of distance $r = U\tau$ along the mean wind directions. The *spatial* structure function in the mean-flow direction $D(r)$ is given by

$$D(r) = \left\langle \left[u\left(\frac{x-r}{U}\right) - u\left(\frac{x}{U}\right)\right]^2 \right\rangle, \quad x = Ut. \tag{39}$$

Thus we may write (38) in the form

$$\langle u^2 \rangle - \ell \langle su \rangle = \frac{1}{2} \int_0^\infty D(r)\, \mathrm{e}^{-r/\ell_\circ} \frac{\mathrm{d}r}{\ell_\circ}, \tag{40}$$

where we have changed integration variable from τ to r and where we have used the relation (25) between the length constant $\ell_\circ$ and the time constant $\tau_\circ$. It is possible to evaluate the integral by using the expression

$$D(r) = \underbrace{\frac{18}{55}\Gamma\left(\frac{1}{3}\right)}_{\simeq 1.3} \alpha\, \varepsilon^{2/3} r^{2/3}, \tag{41}$$

which is valid for small-scale, isotropic turbulence (Lumley & Panofsky 1964, page 84). Here $\alpha \simeq 1.7$ is the Kolmogorov constant and ε the rate of dissipation of the turbulent kinetic energy per unit air mass. Inserting (41) into (40), we get

$$\langle u^2 \rangle - \ell \langle su \rangle = \underbrace{\frac{4\sqrt{3}\pi}{55}}_{\simeq 0.4} \alpha\, (\varepsilon \ell_\circ)^{2/3}. \tag{42}$$

To obtain a practical engineering expression in the case of not too light wind, we use standard relations from the literature about the atmospheric surface layer, e.g. (Panofsky & Dutton 1984, pages 180 and 160). This leads to

$$\langle u^2 \rangle - \ell \langle su \rangle = 0.22\, \langle u^2 \rangle \left(\frac{\ell_\circ}{z}\right)^{2/3}, \tag{43}$$

where z is the height above the ground. Going back to (36) we see that when there is horizontal velocity fluctuations, along and perpendicular to the mean wind direction, the bias becomes

$$\frac{\langle s \rangle}{S} = 0.22 \frac{\langle u^2 \rangle}{U^2} \left(\frac{\ell_\circ}{z}\right)^{2/3} + \frac{\langle v^2 \rangle}{2U^2}. \tag{44}$$

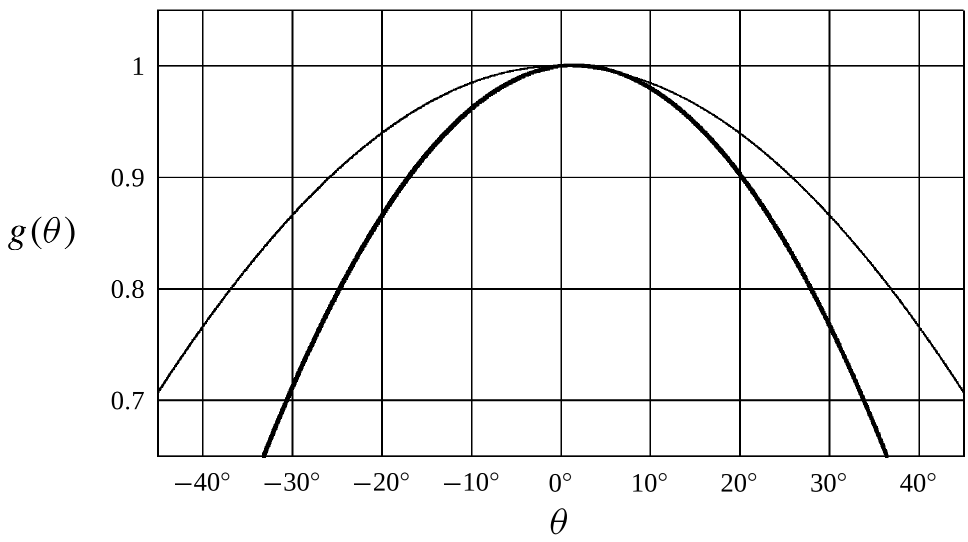

Figure 3: Angular response of the the Risø P2546 cup anemometer (thick line). For comparison the ideal cosine response is shown with a thin line. $(\mu_1, \mu_2) = (0.05, -0.9)$.

We know that $\langle u^2 \rangle$ is about 50% larger than $\langle v^2 \rangle$. However, for most standard cup anemometers $\ell_\circ$ varies between 1 m and 2 m. If we measure at the height $z = 30$ m the value of the factor $(\ell_\circ/z)^{2/3}$ is between 0.1 and 0.2. In other words, the bias from the fluctuations perpendicular to the mean wind is much more important than that from the fluctuations along the the mean wind.

The result (44) would be the final result concerning the bias on the mean-wind speed if the cup anemometer angular response were ideal in the sense that only the wind component perpendicular to the rotor axis exerts a forcing on the rotor. In this case we say that the anemometer has a cosine angular response (see right frame of Fig 2). The angular response $g(\theta)$ is usually not a cosine. In general we have

$$g(\theta) = \cos\theta + \mu_1 \sin\theta + \mu_2(1 - \cos\theta), \tag{45}$$

where μ_1 and μ_2 are dimensionless instrument constants. Figure 3 shows the angular response of the Risø model P2546.

This means that the forcing on the cup rotor is no longer determined by (20), but rather by

$$\widetilde{h}' = \sqrt{\widetilde{h}^2 + \widetilde{w}^2}\, g(\theta). \tag{46}$$

To second order in the perturbing quantities we get, in analogy to (20),

$$\sqrt{\widetilde{h}^2 + \widetilde{w}^2} = \sqrt{(U+u)^2 + v^2 + w^2} \simeq U + u + \frac{v^2 + w^2}{2U}, \tag{47}$$

so that

$$\cos\theta = \frac{\widetilde{h}}{\sqrt{\widetilde{h}^2 + \widetilde{w}^2}} \simeq 1 - \frac{w^2}{2U^2} \tag{48}$$

and

$$\sin\theta = \frac{\widetilde{w}}{\sqrt{\widetilde{h}^2 + \widetilde{w}^2}} \simeq \frac{w}{U} - \frac{uw}{U^2}. \tag{49}$$

Inserting (48) and (49) into (45), the expression for the "apparent" horizontal velocity component becomes

$$\widetilde{h}' = U + (u + \mu_1 w) + \frac{v^2 + \mu_2 w^2}{2U}. \tag{50}$$

In other words, $\widetilde{h}'$ is given by $\widetilde{h}$ if we make the replacements

$$\begin{aligned} u &\rightarrow u + \mu_1 w \\ v^2 &\rightarrow v^2 + \mu_2 w^2. \end{aligned} \tag{51}$$

This implies that (38) is still valid with the replacement

$$\begin{aligned} \left\langle [u(t-\tau) - u(t)]^2 \right\rangle &\rightarrow \left\langle [\{u(t-\tau) + \mu_1 w(t-\tau)\} - \{u(t) + \mu_1 w\}]^2 \right\rangle \\ &= \underbrace{\left\langle [u(t-\tau) - u(t)]^2 \right\rangle}_{=D(U\tau)} + \mu_1^2 \underbrace{\left\langle [w(t-\tau) - w(t)]^2 \right\rangle}_{=\frac{4}{3}D(U\tau)} \\ &\quad + 2\mu_1 \underbrace{\langle [u(t-\tau) - u(t)][w(t-\tau) - w(t)] \rangle}_{=0}. \end{aligned} \tag{52}$$

Under each of the three terms we have given their values according to general rules for isotropic turbulence. These values can be derived by using e.g. Lumley & Panofsky (1964, page 29).

With the replacements (51) the complete bias expression becomes, as a generalization of (44),

$$\frac{\langle s \rangle}{S} = 0.22 \left(1 + \frac{4}{3}\mu_1^2\right) \frac{\langle u^2 \rangle}{U^2} \left(\frac{\ell_\circ}{z}\right)^{2/3} + \frac{\langle v^2 \rangle + \mu_2 \langle w^2 \rangle}{2U^2}. \tag{53}$$

On basis of the considerations above it is also possible to quantify the bias on the variance due to imperfect angular response. It was shown by Kristensen (2000) that the asymmetric response only affects the mean and not higher order moments, in particular the variance. The variance can be determined up to fourth order in s by means of the first-order equation (28) if the angular response were perfect. To include the imperfect angular response we must apply the replacement (51) to (28), which then becomes

$$\dot{s} + \frac{s}{\tau_\circ} = \frac{u + \mu_1 w}{\ell \tau_\circ}. \tag{54}$$

This equation describes a first-order, low-pass filtering of the signal $u(t) + \mu_1 w(t)$ with the time constant $\tau_\circ = \ell_\circ / U$. Since the length scale of the turbulence in a windy atmospheric surface layer is about five times the height z (Kristensen et al. 1989) and $\ell_\circ$ is about 1.8 m, the loss of variance can usually be ignored. This means that the measured variance becomes

$$\sigma^2 \equiv \langle \ell^2 s^2 \rangle = \langle (u + \mu_1 w)^2 \rangle = \langle u^2 \rangle + \mu_1^2 \langle w^2 \rangle + 2\mu_1 \langle uw \rangle. \tag{55}$$

We note that the covariance $\langle uw \rangle$ enters the expression for the measured variance σ^2.

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