

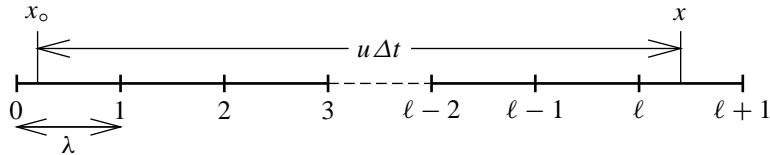
# BIAS ON WIND-SPEED VARIANCE CAUSED BY QUANTIZATION AND SUB-INTERVAL COUNTING OF THE CUP-ANEMOMETER SIGNAL

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## Short Presentation and Conclusion

Every time a column of air of length  $\lambda$  passes through the cup anemometer rotor a pulse is created. For the Risø model P2546,  $\lambda = 0.62$  m and, neglecting in these considerations the bias or starting speed of  $+0.27$  m s<sup>-1</sup>, the count  $N = 7000$  over the time  $T = 10$  min = 600 s corresponds to a *wind way* of 4340 m and a mean-wind speed of about  $U = 7.2$  m s<sup>-1</sup>. The counting does not in general start and stop at the arrival of a pulse and, consequently, the count may be off by 1 corresponding to an uncertainty in the mean-wind speed of 0.001 m s<sup>-1</sup>. This is under all circumstances negligible. However, the pulses are also counted over much smaller time intervals  $\Delta t$ , typically 2 s, and from the  $M = T/\Delta t = 300$  corresponding short *sub-interval* values of the wind speed the variance is calculated over the time  $T$ . In this case the counting uncertainty will in principle give extra variance. It is the purpose in this note to present an estimate of this variance bias. It can be obtained immediately by applying the so-called Sheppard's correction (Sheppard 1898, Wold 1934, Cramér 1946, Kristensen & Kirkegaard 1987). Here we will derive the result directly.



The count  $\ell$  over the time  $\Delta t$  has a mean and a variance which is determined by

$$\langle \ell \rangle = \frac{1}{M} \sum_{m=0}^{M-1} \ell_m \quad (1)$$

and

$$\sigma_\ell^2 = \langle (\ell - \langle \ell \rangle)^2 \rangle = \frac{1}{M} \sum_{m=0}^{M-1} (\ell_m - \langle \ell \rangle)^2, \quad (2)$$

where  $\ell_m$  is count number  $m$  in the period in question.

There is the following relation between the variance,  $\sigma_u^2 = \langle (u - U)^2 \rangle$  of  $u$  and  $\sigma_\ell^2$

$$\frac{\sigma_\ell^2 \lambda^2}{\Delta t^2} = \sigma_u^2 + \frac{\lambda^2}{12 \Delta t^2}. \quad (3)$$

Of course, this equation cannot be valid if the variance  $\sigma_u^2$  is zero or very small. However, Kristensen & Kirkegaard (1987) found that (3) is accurate within 2.5% for a Gaussian process if  $\sigma_u > \lambda/(2\Delta t)$ .

Equation (3) implies that for the Risø model P2546 the variance will be overestimated by the constant amount  $\lambda^2/\Delta t^2/12 \approx 0.008 \text{ m}^2 \text{ s}^{-2}$  if  $\Delta t = 2 \text{ s}$ . The result (3) may be reformulated in terms of the standard deviation SD as follows

$$\text{SD} \equiv \sqrt{\frac{\sigma_\ell^2 \lambda^2}{\Delta t^2}} = \sigma_u \sqrt{1 + \frac{1}{\sigma_u^2} \frac{\lambda^2}{12 \Delta t^2}} \simeq \sigma_u + \frac{1}{\sigma_u} \frac{\lambda^2}{24 \Delta t^2}. \quad (4)$$

For the Risø model P2546 we find that with  $\Delta t = 2 \text{ s}$  and  $\sigma_u = 1 \text{ m s}^{-1}$ , corresponding to the wind speed  $10 \text{ m s}^{-1}$  and the turbulence intensity 0.1, we get the bias correction  $\text{SD} - \sigma_u = 0.004 \text{ m s}^{-1}$ .

In the following we will derive the result (3).

### Derivation of Equation (3)

Obviously we have the simple relation

$$\langle (\ell - \langle \ell \rangle)^2 \rangle = \langle \ell^2 \rangle - \langle \ell \rangle^2 \quad (5)$$

and we proceed by determining  $\langle \ell \rangle$  and  $\langle \ell^2 \rangle$  on basis of the probability density function for the wind speed  $u$  averaged over the time interval  $\Delta t$ . First we derive the probability  $P[\ell]$  for the count  $\ell$  from the probability density  $\phi_o(y)$  for the wind way  $y = u\Delta t$  which, since  $\Delta t$  is kept constant, is equal to the probability density function for  $u$  divided by  $\Delta t$ . Inspecting the sketch, we see that  $P[\ell]$  is the probability that the beginning of the wind way  $0 \leq x_o \leq \lambda$  falls in the first  $\lambda$ -interval and that the end  $x = x_o + y$  falls in the  $\ell$ 'th  $\lambda$ -interval. In other words

$$P[\ell] = \int_0^\lambda \frac{dx_o}{\lambda} \int_{\ell\lambda}^{(\ell+1)\lambda} \phi_o(x - x_o) dx. \quad (6)$$

Introducing for convenience

$$\phi(x) = \int_0^\lambda \phi_o(x - x_o) \frac{dx_o}{\lambda}, \quad (7)$$

we can rewrite (6) in the form

$$P[\ell] = \int_{\ell\lambda}^{(\ell+1)\lambda} \phi(x) dx. \quad (8)$$

Since

$$\int_0^\infty \phi(x) dx = \int_0^\lambda \frac{dx_o}{\lambda} \int_0^\infty \phi_o(x - x_o) dx = 1, \quad (9)$$

we see immediately that

$$\sum_{\ell=0}^{\infty} P[\ell] = 1. \quad (10)$$

The mean of  $\ell$  and  $\ell^2$  become

$$\begin{aligned} \langle \ell \rangle &= \sum_{\ell=0}^{\infty} \ell P[\ell] = \sum_{\ell=0}^{\infty} \ell \int_{\ell\lambda}^{(\ell+1)\lambda} \phi(x) dx = \sum_{\ell=0}^{\infty} \ell \int_{\ell\lambda}^{\infty} \phi(x) dx - \sum_{\ell=0}^{\infty} \ell \int_{(\ell+1)\lambda}^{\infty} \phi(x) dx \\ &= \sum_{\ell=1}^{\infty} \ell \int_{\ell\lambda}^{\infty} \phi(x) dx - \sum_{\ell=1}^{\infty} (\ell - 1) \int_{\ell\lambda}^{\infty} \phi(x) dx = \sum_{\ell=0}^{\infty} \int_{(\ell+1)\lambda}^{\infty} \phi(x) dx \end{aligned} \quad (11)$$

and

$$\begin{aligned}
\langle \ell^2 \rangle &= \sum_{\ell=0}^{\infty} \ell^2 P[\ell] = \sum_{\ell=0}^{\infty} \ell^2 \int_{\ell\lambda}^{(\ell+1)\lambda} \phi(x) dx = \sum_{\ell=0}^{\infty} \ell^2 \int_{\ell\lambda}^{\infty} \phi(x) dx - \sum_{\ell=0}^{\infty} \ell^2 \int_{(\ell+1)\lambda}^{\infty} \phi(x) dx \\
&= \sum_{\ell=1}^{\infty} \ell^2 \int_{\ell\lambda}^{\infty} \phi(x) dx - \sum_{\ell=1}^{\infty} (\ell-1)^2 \int_{\ell\lambda}^{\infty} \phi(x) dx = \sum_{\ell=0}^{\infty} (2\ell+1) \int_{(\ell+1)\lambda}^{\infty} \phi(x) dx. \quad (12)
\end{aligned}$$

To proceed we need an approximate relation between infinite summations and infinite integrals. We start with the identity

$$\int_0^{\infty} f(x) dx = \sum_{\ell=0}^{\infty} \int_{\ell\lambda}^{(\ell+1)\lambda} f(x) dx, \quad (13)$$

where  $f(x)$  can be any integrable function for which  $f'(\infty) = 0$ . Taylor expanding of  $f(x)$  from the midpoint  $(\ell + 1/2)\lambda$  yields

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}((\ell + 1/2)\lambda)}{n!} (x - (\ell + 1/2)\lambda)^n, \quad (14)$$

where  $f^{(n)}(x)$  is the  $n$ 'th derivative of  $f(x)$ . Thus

$$\begin{aligned}
\int_{\ell\lambda}^{(\ell+1)\lambda} f(x) dx &= \int_{\ell\lambda}^{(\ell+1)\lambda} dx \sum_{n=0}^{\infty} \frac{f^{(n)}((\ell + 1/2)\lambda)}{n!} (x - (\ell + 1/2)\lambda)^n \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}((\ell + 1/2)\lambda)}{n!} \int_{-\lambda/2}^{\lambda/2} s^n ds = \sum_{n=0}^{\infty} \frac{f^{(n)}((\ell + 1/2)\lambda)}{(n+1)!} \left\{ \left(\frac{\lambda}{2}\right)^{n+1} - \left(-\frac{\lambda}{2}\right)^{n+1} \right\} \\
&= \sum_{n=0}^{\infty} \frac{f^{(2n)}((\ell + 1/2)\lambda)}{2^{2n} (2n+1)!} \lambda^{2n+1}. \quad (15)
\end{aligned}$$

Inserting in (13), we obtain

$$\int_0^{\infty} f(x) dx = \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2^{2n}(2n+1)!} \sum_{\ell=0}^{\infty} f^{(2n)}((\ell + 1/2)\lambda). \quad (16)$$

This relation is one form of the Euler-Maclaurin sum formula. Since  $f(x)$  can be any function it also applies to its second derivative  $f''(x)$ , i.e.

$$\int_0^{\infty} f''(x) dx = \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2^{2n}(2n+1)!} \sum_{\ell=0}^{\infty} f^{(2n+2)}((\ell + 1/2)\lambda). \quad (17)$$

Truncating (16) after the second term and (17) after the first, we have

$$\int_0^{\infty} f(x) dx \simeq \lambda \sum_{\ell=0}^{\infty} f((\ell + 1/2)\lambda) + \frac{\lambda^3}{24} \sum_{\ell=0}^{\infty} f''((\ell + 1/2)\lambda) \quad (18)$$

and

$$\int_0^{\infty} f''(x) dx \simeq \lambda \sum_{\ell=0}^{\infty} f''((\ell + 1/2)\lambda). \quad (19)$$

Combining these two equations, we get

$$\int_0^{\infty} f(x) dx \simeq \lambda \sum_{\ell=0}^{\infty} f((\ell + 1/2)\lambda) + \frac{\lambda^2}{24} \int_0^{\infty} f''(x) dx = \lambda \sum_{\ell=0}^{\infty} f((\ell + 1/2)\lambda) - \frac{\lambda^2}{24} f'(0) \quad (20)$$

or

$$\sum_{\ell=0}^{\infty} f((\ell + 1/2)\lambda) \simeq \frac{1}{\lambda} \int_0^{\infty} f(x) dx + \frac{\lambda}{24} f'(0). \quad (21)$$

We can now determine  $\langle \ell \rangle$  and  $\langle \ell^2 \rangle$ . We start with (11).

$$\begin{aligned}
\langle \ell \rangle &= \sum_{\ell=0}^{\infty} \underbrace{\int_{(\ell+1)\lambda}^{\infty} \phi(s) ds}_{f((\ell+1/2)\lambda)} \simeq \frac{1}{\lambda} \int_0^{\infty} dx \int_{x+\lambda/2}^{\infty} \phi(s) ds - \frac{\lambda}{24} \phi(\lambda/2) \\
&= \frac{1}{\lambda} \int_0^{\infty} x \phi(x + \lambda/2) dx - \frac{\lambda}{24} \phi(\lambda/2) \simeq \frac{\langle u \rangle \Delta t}{\lambda}.
\end{aligned} \tag{22}$$

$$\begin{aligned}
\langle \ell^2 \rangle &= \sum_{\ell=0}^{\infty} (2\ell + 1) \underbrace{\int_{(\ell+1)\lambda}^{\infty} \phi(s) ds}_{f((\ell+1/2)\lambda)} \simeq \frac{1}{\lambda^2} \int_0^{\infty} 2x dx \int_{x+\lambda/2}^{\infty} \phi(s) ds + \frac{1}{12} \int_{\lambda/2}^{\infty} \phi(s) ds \\
&= \frac{1}{\lambda^2} \int_0^{\infty} x^2 \phi(x + \lambda/2) dx + \frac{1}{12} \int_{\lambda/2}^{\infty} \phi(s) ds \simeq \frac{\langle u^2 \rangle \Delta t^2}{\lambda^2} + \frac{1}{12}.
\end{aligned} \tag{23}$$

The last two equations imply that the measured variance becomes

$$\langle (\ell - \langle \ell \rangle)^2 \lambda^2 \rangle \simeq \langle (u - \langle u \rangle)^2 \rangle \Delta t^2 + \frac{\lambda^2}{12}. \tag{24}$$

## References

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